A practical test for positive definiteness that does not require explicit calculation of the eigenvalues is the principal minor test. The kth leading principal minor is the determinant formed by deleting the last $n-k$ rows and columns of the matrix. A necessary and sufficient condition that a symmetric $n \times n$ matrix be positive definite is that all n leading principal minors Δ_k are positive.[1]

However, the analogous statement that a necessary and sufficient condition that a matrix be positive semidefinite is that all n leading principal minors are nonnegative is not true, yet this statement is found in some textbooks and reference books. Greenwood[2] states that if one or more of the leading principal minors are zero, but none are negative, then the matrix is positive semidefinite. This is disproved by the examples in this Note. Brogan[3] states that a test for positive semidefiniteness is that all of the leading principal minors are nonnegative, which incorrectly implies that if this test is satisfied, the matrix is positive semidefinite. Wiberg[4] provides the analogous condition for a Hermitian matrix, namely, that it is positive semidefinite, if, and only if, all of the leading principal minors are nonnegative.

As a trivial example consider the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

Both leading principal minors are zero and hence nonnegative, but the matrix is obviously not positive semidefinite. One eigenvalue is zero, the other is -1.

Another example is the 3×3 symmetric matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{bmatrix} \tag{2}$$

The leading principal minors are nonnegative ($\Delta_1 = 1$, $\Delta_2 = \Delta_3 = 0$), but the matrix is not positive semidefinite. This can be verified by calculating the value of the quadratic form $q = x^T A x$, where $x^T = [x_1 x_2 x_3]$ as $q = (x_1 + x_2 + x_3)^2 + (a-1)x_3^2$. By inspection, one can guarantee $q \geq 0$ for all x only if $a \geq 1$. For example, if $a < 1$, vectors in the plane $x_1 + x_2 + x_3 = 0$ yield $q < 0$. Note, however, that the values of the leading principal minors are independent of the value a.

It is easily shown that all of the leading principal minors of a positive semidefinite matrix are nonnegative, for example, by considering vectors x having the last $n-k$ elements equal to zero. Thus, the condition that $\Delta_k \geq 0$ is apparently a necessary but not a sufficient condition for positive semidefiniteness. In order to state a condition that is also sufficient, one must consider principal minors D_k formed by deleting *any* $n-k$ rows and corresponding columns. The correct necessary and sufficient condition is that all possible principal minors are nonnegative.

As an example, consider the matrix in Eq. (1). If one calculates the principal minors D_k formed by deleting the first rather than last $n-k$ rows and columns, one finds that $D_1 = -1$ and $D_2 = 0$, which clearly violates the condition. In the same manner, the principal minors of the matrix in Eq. (2) are $D_1 = a$, $D_2 = a-1$, and $D_3 = 0$, satisfying the condition only if $a \geq 1$.

For an $n \times n$ matrix that are $\binom{n}{k}$ principal minors D_k, where $\binom{n}{k}$ is the binomial coefficient $n!/(n-k)!k!$. The total number of principal minors is then

$$m = \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1 \tag{3}$$

Thus, $m = 7$ for $n = 3$, and $m = 15$ for $n = 4$, indicating that appreciably more computation may be required to determine the value of all possible principal minors. However, if all n leading principal minors of a matrix are nonnegative, with one or more having a zero value, all m principal minors must be shown to be nonnegative in order to guarantee that the matrix is positive semidefinite.

References

[1] Hildebrand, F.B., *Methods of Applied Mathematics,* Prentice-Hall, Englewood Cliffs, NJ, 1961, p. 50.

[2] Greenwood, D.T., *Principles of Dynamics,* Prentice-Hall, Englewood Cliffs, NJ, 1965, p. 452.

[3] Brogan, W.L., *Modern Control Theory,* Quantum Publishers, New York, 1974, p. 134.

[4] Wiberg, D.M., *State Space and Linear Systems,* Schaum Outline Series, McGraw-Hill Book Co., New York, 1971, p. 77.

Gravity Gradient Torque for an Arbitrary Potential Function

David R. Glandorf*
Lockheed Engineering
and Management Services Company
Houston, Texas

Nomenclature

C	$= C_a$ to C_s transformation matrix
C_a	= attracting body coordinate system
C_s	= spacecraft coordinate system
dm	= differential mass
F_g	= gravitational force
$g \triangleq (g_x, g_y, g_z)^T$	= components of g in C_a
g	= gravitational acceleration
G	= gravity gradient dyadic
G	= gravity gradient matrix in C_a
I	$= 3 \times 3$ identity matrix
I_x, I_y, I_z	= spacecraft moments of inertia
I_{xy}, I_{xz}, I_{yz}	= spacecraft products of inertia
J	= spacecraft inertia dyadic
J	= components of J in C_s
m	= spacecraft mass
r	= magnitude of R
$R \triangleq (x,y,z)^T$	= components of R in C_a
R	= location of dm
R_c	= location of spacecraft mass center
$U \triangleq (U_x, U_y, U_z)^T$	= components of U in C_a
U	= unit vector in the direction of R
λ_i	= eigenvalues of G, $i = 1,2,3$
μ	= gravitational parameter of attracting body
$\rho \triangleq (\rho_x, \rho_y, \rho_z)$	= components of ρ in C_s
ρ	= location of dm relative to spacecraft mass center
$\tau_{gg} \triangleq (\tau_{ggx}, \tau_{ggy}, \tau_{ggz})^T$	= components of τ_{gg} in C_s
τ_g	= gravitational moment
τ_{gg}	= gravity gradient torque
ϕ	= gravitational potential
ψ_i	= components of ψ_i in C_s, $i = 1,2,3$
$\hat{\psi}_i$	= components of ψ_i in C_a, $i = 1,2,3$
ψ_i	= eigenvectors of G, $i = 1,2,3$

Received April 1, 1985; revision received July 16, 1985.

*Advanced Systems Engineering Specialist. Member AIAA.

Superscript

T = transpose

Introduction

IT is well known that gravitational torques acting on spacecraft play an important role in the spacecraft's rotational dynamics and stability.[1-7] Various formulations of the gravitational torque have been proposed, but they are all based on the distribution of the gravitational potential or acceleration throughout the spacecraft. Since no gravitational torque exists in a uniform gravitational field, the gravitational torque has commonly been referred to as the gravity gradient torque. Roberson[1] developed a formulation for a potential function based on an oblate spheroid central body. Nidey[2] developed a simpler form of a solution for an inverse square gravitational acceleration field. Roberson[3] derived an alternate form of Nidey's result in dyadic notation. Nidey[2,4] also discusses the computation of the average gravity gradient torque on a satellite in a circular orbit. Hultquist[5] determines the gravity gradient torque effect on spacecraft in both circular and elliptic orbits. Holland and Sperling[6] study the combined effects of gravity gradient and magnetic torques on the rotational motion of a spacecraft. A brief discussion of gravity gradient torques is given by Spence.[7] Except for Roberson's work,[1] the above investigations were based on an inverse square law gravitational acceleration field. While this is undoubtedly the main contributor to the gravity gradient torque, it would be helpful to have available a simple method of computing the gravity gradient torque for more complex potential functions. This would permit investigation of the relative effects of higher-order terms in the potential function and allow computation of gravity gradient torques consistent with the gravitational acceleration for a given potential function. A simple formula is derived for computing the gravity gradient torque on a vehicle in terms of the eigenvalues and eigenvectors of the gravity gradient matrix and the inertia properties of the vehicle. This formulation is applicable to any potential function for which the gravity gradient matrix can be computed for a given point in the potential field.

Formulation

The gravitational force and moment on the spacecraft are given by

$$F_g = \int_m g(R)dm \qquad \tau_g = \int_m \rho \times g(R)dm \tag{1}$$

where the integration is over the mass of the spacecraft and

$$g(R) = \nabla\phi(R) \tag{2}$$

To first order $g(R)$ can be expanded about R_c as

$$g(R) = g(R_c) + \mathrm{G}(R_c)\cdot\rho \tag{3}$$

Substitution of Eq. (3) into Eqs. (1) gives

$$F_g = g(R_c)\int_m dm + \mathrm{G}(R_c)\cdot\int_m \rho dm \tag{4a}$$

$$\tau_g = \int_m \rho\times\mathrm{G}(R_c)\cdot\rho dm - g(R_c)\times\int_m \rho dm \tag{4b}$$

But $\int_m \rho dm = 0$, so the second terms in the expressions for F_g and τ_g both vanish. Furthermore, $\int_m dm = m$, so the gravitational force becomes simply $F_g = mg(R_c)$. The first term in the expression for τ_g is called the gravity gradient torque because it depends on the gradient of the gravitational acceleration. We thus have

$$\tau_{gg} \triangleq \int_m \rho\times\mathrm{G}(R_c)\cdot\rho dm \tag{5}$$

and the evaluation of this integral is our primary concern.

Solution

A minor inconvenience in the evaluation of the integral in Eq. (5) is that ρ is most conveniently defined in terms of components in C_s while $\mathrm{G}(R_c)$ is most conveniently defined in terms of components in C_a. Since the rotational dynamics are most often expressed in C_s, we rewrite Eq. (5) in C_s as

$$\tau_{gg} = \int_m \tilde{\rho}CG(R_c)C^T\rho dm \tag{6}$$

where

$$\tilde{\rho} \triangleq \begin{bmatrix} 0 & -\rho_z & \rho_y \\ \rho_z & 0 & -\rho_x \\ -\rho_y & \rho_x & 0 \end{bmatrix} \tag{7}$$

$$G(R) = \begin{bmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{bmatrix} \tag{8}$$

and $g_{ij} \triangleq \partial g_i/\partial j\,(i,j=x,y,z)$. From Eq. (2) we have $g_i = \phi_i \triangleq \partial\phi/\partial i$, so $g_{ij} = \phi_{ij} \triangleq \partial^2\phi/\partial i\partial j$; and since $\phi_{ij} = \phi_{ji}$, it follows that $G(R)$ is symmetric. Therefore, $G(R)$ has a set of three real eigenvalues and a corresponding set of three real orthonormal eigenvectors. If two or more of the eigenvalues are identical, the corresponding eigenvectors are not unique, but this does not present a problem. The gravity gradient matrix can be expressed in terms of its eigenvalues and eigenvectors as

$$G(R) = \Psi\Lambda\Psi^T \tag{9}$$

where

$$\Psi \triangleq [\hat{\psi}_1\hat{\psi}_2\hat{\psi}_3] \qquad \Lambda \triangleq \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3) \tag{10}$$

Substitution of Eqs. (10) into Eq. (9) gives

$$G(R) = \sum_{i=1}^{3}\lambda_i\hat{\psi}_i\hat{\psi}_i^T \tag{11}$$

This expansion of $G(R)$ is the key to the general solution of Eq. (6), which with the relation $\psi_i = C\hat{\psi}_i$ can now be written

$$\tau_{gg} = \sum_{i=1}^{3}\lambda_i\int_m \tilde{\rho}\psi_i\psi_i^T\rho dm \tag{12}$$

But $\tilde{\rho}\psi_i$ and $\psi_i^T\rho$ are just alternate forms of the vector cross-product $(\rho\times\psi_i)$ and dot-product $(\rho\cdot\psi_i)$, respectively, so we have

$$\tilde{\rho}\psi_i = -\tilde{\psi}_i\rho \qquad \psi_i^T\rho = \rho^T\psi_i \tag{13}$$

where $\tilde{\psi}_i$ is a skew-symmetric matrix formed from the components of ψ_i analogous to the definition of $\tilde{\rho}$ in Eq. (7). Substitution of Eqs. (13) into Eq. (12) and recognition that $\tilde{\psi}_i$ and ψ_i can be factored out of the integral gives

$$\tau_{gg} = \sum_{i=1}^{3}\lambda_i\tilde{\psi}_i\left[\int_m(-\rho\rho^T)dm\right]\psi_i \tag{14}$$

It can also be shown that $\rho\rho^T = \tilde{\rho}\tilde{\rho} + (\rho^T\rho)I$, so Eq. (14) becomes

$$\tau_{gg} = \sum_{i=1}^{3} \lambda_i \tilde{\psi}_i \left[\int_m (-\tilde{\rho}\tilde{\rho})dm\right]\psi_i - \left[\int_m \rho^T\rho dm\right] \sum_{i=1}^{3} \lambda_i \tilde{\psi}_i \psi_i$$
$$= \sum_{i=1}^{3} \lambda_i \tilde{\psi}_i \left[\int_m (-\tilde{\rho}\tilde{\rho})dm\right]\psi_i \tag{15}$$

where use has been made of the relation $\tilde{\psi}_i\psi_i = \psi_i \times \psi_i = 0$. Expansion of the remaining integral in the above equation leads to

$$\int_m (-\tilde{\rho}\tilde{\rho})dm = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \triangleq J \tag{16}$$

where the moments and products of inertia of the spacecraft are assumed to be known. Substitution of Eq. (16) into Eq. (15) provides the final general formula for the gravity gradient torque

$$\tau_{gg} = \sum_{i=1}^{3} \lambda_i \tilde{\psi}_i J \psi_i \tag{17}$$

Implementation of this result requires: 1) calculation of the gravity gradient matrix, 2) determination of its eigenvalues and eigenvectors, 3) transformation of the eigenvectors to the spacecraft coordinate system, and 4) evaluation of the summation in Eq. (17). Conceptually, these steps may be done analytically or numerically. However, since the gravity gradient matrix depends on the position of the spacecraft relative to the attracting body, analytic solutions for the eigenvalues and eigenvectors are expected to be difficult to find except perhaps for very simple potential functions. A numerical solution would then be required at each spacecraft position of interest.

Analytic Example

As an example of an analytic application of the preceding results, we consider a potential function which is inversely proportional to the distance from the origin $\phi(R) = \mu/r$ where $r = (x^2 + y^2 + z^2)^{1/2}$ and

$$G(R) = -(\mu/r^3)(I - 3RR^T/r^2) \tag{18}$$

The eigenvalues are determined from $\det[\lambda I - G(R)] = 0$ or

$$\det(\sigma I - UU^T) = 0 \tag{19}$$

where

$$\sigma \triangleq (r^3/3\mu)(\lambda + \mu/r^3) \tag{20a}$$

$$U = R/r \tag{20b}$$

The three solutions to Eq. (19) are $\sigma = 0, 0, 1$, while from Eq. (20a) the corresponding eigenvalues of $G(R)$ are

$$\lambda_1 = \lambda_2 = -\mu/r^3 \qquad \lambda_3 = 2\mu/r^3 \tag{21}$$

The corresponding eigenvectors are found from $[\lambda_i I - G(R)]\hat{\psi}_i = 0$ or

$$[(\lambda_i + \mu/r^3)I - (3\mu/r^3)UU^T]\hat{\psi}_i = 0 \tag{22}$$

For λ_1 and λ_2, this becomes $UU^T\hat{\psi}_i = 0$ or simply $U^T\hat{\psi}_i = 0$, so ψ_1 and ψ_2 are two unit vectors orthogonal to $\boldsymbol{R}$ and each other. For λ_3, Eq. (22) becomes $\hat{\psi}_3 - UU^T\hat{\psi}_3 = 0$ which has a solution $\hat{\psi}_3 = U$, so ψ_3 is a unit vector in the direction of $\boldsymbol{R}$. Substitution of Eqs. (21) into Eq. (17) and a little manipulation of the result now gives

$$\tau_{gg} = -\frac{\mu}{r^3}\sum_{i=1}^{3} \tilde{\psi}_i J \psi_i + \frac{3\mu}{r^3}\tilde{\psi}_3 J \psi_3 \tag{23}$$

Examination of the development of Eq. (17) from Eq. (14) reveals that the λ_i are insignificant in the intermediate steps. By analogy we then have

$$\sum_{i=1}^{3} \tilde{\psi}_i J \psi_i = -\int_m \sum_{i=1}^{3} \tilde{\psi}_i \rho\rho^T \psi_i dm \tag{24}$$

Since the ψ_i are mutually orthogonal unit vectors, they form a basis, and ρ can be written as

$$\rho = \rho_1\psi_1 + \rho_2\psi_2 + \rho_3\psi_3 \tag{25}$$

where

$$\rho_i = \rho^T\psi_i \tag{26}$$

From Eq. (25), we have $\tilde{\psi}_i\rho = \rho_1\tilde{\psi}_i\psi_1 + \rho_2\tilde{\psi}_i\psi_2 + \rho_3\tilde{\psi}_i\psi_3$ which can be combined with Eq. (26) to give $\tilde{\psi}_i\rho\rho^T\psi_i = \rho_i\rho_1\tilde{\psi}_i\psi_1 + \rho_i\rho_2\tilde{\psi}_i\psi_2 + \rho_i\rho_3\tilde{\psi}_i\psi_3$ from which it is readily shown that

$$\sum_{i=1}^{3} \tilde{\psi}_i \rho\rho^T \psi_i = 0 \tag{27}$$

and Eq. (24) reduces to

$$\sum_{i=1}^{3} \tilde{\psi}_i J \psi_i = 0 \tag{28}$$

Substitution of this result into Eq. (23) using Eq. (20b) and the solution for ψ_3 yields

$$\tau_{gg} = (3\mu/r^5)\tilde{R}JR \tag{29}$$

or in vector-dyadic notation

$$\tau_{gg} = (3\mu/r^5)\boldsymbol{R} \times \mathbf{J} \cdot \boldsymbol{R} \tag{30}$$

where R in Eq. (29) denotes the coordinates of the spacecraft in the spacecraft axes system. This result is consistent with the result derived directly from Eq. (6) using Eq. (18) without expressing $G(R)$ in terms of its eigenvalues and eigenvectors.[7]

Conclusions

A simple concise formula has been presented for the computation of the gravity gradient torque on a spacecraft in an arbitrary potential function field. The formula requires only the eigenvalues and eigenvectors of the gravity gradient matrix evaluated at the center of mass of the spacecraft, and the corresponding spacecraft inertia properties. The result is probably of more academic than practical interest because the effects of higher-order terms in the potential function on the gravity gradient torque are not expected to be significant for most applications.

Acknowledgment

This work was supported by NASA Lyndon B. Johnson Space Center under contract NAS 9-15800.

References

[1]Roberson, R.E., "Gravitational Torque on a Satellite Vehicle," *Journal of the Franklin Institute,* Vol. 265, Jan. 1958, pp. 13-22.

[2]Nidey, R.A., "Gravitational Torque on a Satellite of Arbitrary Shape," *ARS Journal,* Vol. 30, Feb. 1960, pp. 203-204.

[3]Roberson, R.E., "Alternate Form of a Result by Nidey," *ARS Journal,* Vol. 31, Sept. 1961, pp. 1292-1293.

[4]Nidey, R.A., "Secular Gravitational Torque on a Satellite in a Circular Orbit," *ARS Journal,* Vol. 31, 1961, p. 1032.

[5]Hultquist, P.F., "Gravitational Torque Impulse on a Stabilized Satellite," *ARS Journal,* Vol. 31, Nov. 1961, pp. 1506-1509.

[6]Holland, R.L. and Sperling, H.J., "A First-Order Theory for the Rotational Motion of a Triaxial Rigid Body Orbiting an Oblate Primary," *Astonomical Journal,* Vol. 74, April 1969, pp. 490-496.

[7]Spence, C.B. Jr., "Environmental Torques," in *Spacecraft Attitude Determination and Control,* edited by James R. Wertz, D. Reidel Pub. Co., Boston, 1978, pp. 566-570.